THE

MATHEMATICAL GAZETTE

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc., F.R.S.
AND
PROF. E. T. WHITTAKER, Sc.D., F.R.S.

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STRAY NOTES ON ELEMENTARY GEOMETRY.

BY T. J. I'A BROMWICH, F.R.S.

SOME twenty-five years ago I was in the unpleasant position of "professing" Geometry, with (roughly) six weeks start of the class; it is no disparagement to the abilities of the Galway Honour Students to confess (at this late hour) that—however hard put to it—I never had reason to admit greater ignorance than that of the class. Amongst other things I was obliged to learn that there is a certain projective relation between the 27 lines on a cubic surface and the 28 bitangents to a (general) plane quartic; this fact impressed me so much that it remains (also a general idea of where a proof can be found) even now, when my memory has discarded many things which are (perhaps) more vital.

However, at an earlier stage the class asked me to attempt to revise some general properties of conics, which I had been forgetting (with greater success than usual) since my degree in 1895. And in solving a certain example on a hyperbola I attempted to use a property of the general conic, which (according to my solution) proved fallacious. This paradox was recently brought to my memory; and it may prove interesting for readers of the *Gazette*.

We start with two chords of a general conic OPQ and ORS; then (as proved by Salmon and other writers) $OP.OQ/OR.OS$ depends only on the *directions* of the chords and not on the point O. Consider next the special case in which S tends to infinity (so that the conic must be either a parabola or a hyperbola). Then we should anticipate that

$$O_1S_1/O_2S_2 \to 1$$

for two positions of O_1, O_2.

Thus the general theorem would appear to lead to the conclusion that $OP.OQ/OR = \text{const.}$ if PQ is in any fixed direction, and OR is parallel to one of the asymptotes for a hyperbola (or to the axis for a parabola).

By applying this version to a few easy cases I soon disproved it for a hyperbola, although it is always true for a parabola. I have never seen an entirely convincing discussion of the fallacy by pure geometry; but (as we shall see below) the error is introduced through the assumption that

$$O_1S_1/O_2S_2 \to 1.$$

To deal with the hyperbola by analytical methods, let us take oblique axes parallel to the asymptotes—so that the equation to the curve is

$$xy = c.$$

Then let (x_0, y_0) be O, and l, m direction-ratios for the line OPQ (fixed in direction). Then OP, OQ are roots of the quadratic in r,

$$(x_0+rl)(y_0+rm)=c,$$

so that

$$OP.OQ=(x_0y_0-c)/lm.$$

For a line parallel to x (say) we have $y=y_0$, which cuts the hyperbola in the point given by $x=c/y_0$, so that

$$OR=x_0-c/y_0=(x_0y_0-c)/y_0.$$

Thus

$$\frac{OP.OQ}{OR}=\frac{y_0}{lm},$$

and y_0 is proportional to ϖ the distance from O to the other asymptote.

Thus the corrected form of the theorem

$$\frac{O_1P_1.O_1Q_1}{O_1R_1.\varpi_1}=\frac{O_2P_2.O_2Q_2}{O_2R_2.\varpi_2}.$$

We can connect this result with the general theorem, from which we started, by observing that χ (the length of the chord RS) is equal to the difference between the roots of the quadratic in r; and so

$$\chi^2=\left(\frac{x_0}{l}+\frac{y_0}{m}\right)^2-4\frac{x_0y_0-c}{lm}=\left(\frac{y_0}{m}-\frac{x_0}{l}\right)^2+\frac{4c}{lm}.$$

Then, if the chord tends to the direction of the axis of x, so that $l\to 1$, $m\to 0$, (the point O being kept fixed), it is evident that $\chi\to\infty$ in such a way that

$$m\chi\to y_0.$$

Consequently if $O_1R_1S_1$, $O_2R_2S_2$ are two chords which are kept parallel to each other, and tend in that manner towards the direction of the x-asymptote, we shall have

$$R_1S_1/R_2S_2\to y_1/y_2=\varpi_1/\varpi_2.$$

But O_1R_1, O_2R_2 tend to finite limits, and so we have also

$$O_1S_1/O_2S_2\to\varpi_1/\varpi_2$$

instead of the result which was anticipated.

A corresponding calculation for the parabola gives (without any difficulty)

$$OP.OQ/OR=\text{const.},$$

and we can prove, in a similar way, that here $O_1S_1/O_2S_2\to 1$.

Cambridge, 16th June, 1928. T. J. I'A. BROMWICH.

P.S.—Since the above was written my attention was drawn by a former pupil to Prof. E. H. Neville's note, No. 879, in which exactly the same question occurs; it seems, however, worth while to let my solution stand (being at least twenty-five years older). I proposed the fallacy as a question at St. John's College, Cambridge, in June 1908.

Glancing at this page, I observed note No. 880, by Mr. A. J. H. Morrell; and it then struck me that one could use complex coordinates (originally used in such questions by Prof. Frank Morley of Johns Hopkins—see for some examples, Ch. X. of either edition of my book on *Infinite Series*). Thus, let (ξ, η) be ordinary rectangular Cartesian coordinates; and use instead

$$x=\xi+\iota\eta,\quad y=\xi-\iota\eta.$$

For example, the usual R.H. $\xi^2-\eta^2=a^2$ is replaced by $x^2+y^2=2a^2$.

The process of inversion is given by

$$x_1=k^2/y,\quad y_1=k^2/x,$$

and so *the inverse to the* R.H. *is*

$$\frac{1}{x^2}+\frac{1}{y^2}=\frac{2a^2}{k^4}.$$

The pedal of the R.H. is found by taking first the tangent at (x_0, y_0), namely,

$$xx_0+yy_0=2a^2,$$

and the perpendicular from the origin on the tangent is accordingly

$$xx_0-yy_0=0.$$

Thus the foot of the perpendicular is given by

$$xx_0=yy_0=a^2,$$

and the equation to the pedal is therefore

$$\frac{1}{x^2}+\frac{1}{y^2}=\frac{2}{a^2},$$

which is the same as the inverse, provided that we take $k=a$.

The pedal of the lemniscate is found from its tangent

$$\frac{x}{x_2^3}+\frac{y}{y_2^3}=\frac{1}{x_2^2}+\frac{1}{y_2^2}=\frac{2}{a^2},$$

and the foot of the perpendicular on this line is given by

$$\frac{x}{x_2^3}=\frac{y}{y_2^3}=\frac{1}{a^2}.$$

Thus the equation to the pedal of the lemniscate is

$$\left(\frac{a}{x}\right)^{\frac{1}{3}}+\left(\frac{a}{y}\right)^{\frac{1}{3}}=2,$$

but a more compact and convenient form is given by using $t\ (=x_2/y_2)$ as a parameter (so that $|t|=1$); this gives at once

$$\frac{2x_2^3}{a^3}=1+t^3,\quad \frac{2y_2^3}{a^3}=1+\frac{1}{t^3}.$$

Thus
$$\frac{x}{a}=\left(\frac{x_2}{a}\right)^3=\left(\frac{1+t^3}{2}\right)^{\frac{1}{3}},\quad \frac{y}{a}=\left(\frac{y_2}{a}\right)^3=\frac{1}{t^3}\left(\frac{1+t^3}{2}\right)^{\frac{1}{3}}.$$

If we write now $x=re^{i\theta}$, $y=re^{-i\theta}$, $t=e^{i\omega}$ so that r, θ are the usual polar coordinates of a point on the pedal, then

$$e^{2i\theta}=x/y=e^{3i\omega}\quad\text{or}\quad \omega=\tfrac{2}{3}\theta$$

and
$$r^2=xy=a^2(\tfrac{1}{2}(t+1/t))^3=a^2\cos^3\omega.$$

Thus the polar equation of the pedal of the lemniscate is given by

$$(r/a)^{\frac{2}{3}}=\cos\tfrac{2}{3}\theta.$$

Passing on still further I observed also Prof. E. H. Smart's note No. 881; part of the proof can be arranged very neatly by the aid of these methods.

Suppose then that we fix our attention on the stage in Prof. Smart's solution at which the (variable) points A, B, C are on a fixed circle; this circle may be taken as $|x|=1$, and $x=a$, b, c may be the values corresponding to A, B, C, so that we have also $y=1/a$, $1/b$, $1/c$ at these points. Then G has the coordinates

$$x=\tfrac{1}{3}(a+b+c)\quad y=\tfrac{1}{3}(1/a+1/b+1/c),$$

and if G lies on a concentric circle $xy=\sigma^2$ we have

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=9\sigma^2.$$

Now QR is the line joining the point $x=\frac{1}{2}(a+b)$, $y=\frac{1}{2}(1/a+1/b)$, to the point $x=\frac{1}{2}(a+c)$, $y=\frac{1}{2}(1/a+1/c)$ and so QR has the equation

$$2x-a+bc\left(2y-\frac{1}{a}\right)=b+c,$$

or
$$2a(x+bcy)=(a+b)(a+c).$$

Now P is the point $x=\frac{1}{2}(b+c)$, $y=\frac{1}{2}(1/b+1/c)$, so that the polar line of P with respect to a third concentric circle $xy=\tau^2$ has the equation

$$\frac{x}{2}\left(\frac{1}{b}+\frac{1}{c}\right)+\frac{y}{2}(b+c)=2\tau^2,$$

or

$$(b+c)(x+bcy)=4bc\tau^2.$$

Comparing this equation with that already found for QR, we see that the two lines are identical provided that

$$(b+c)(c+a)(a+b)=8abc\tau^2,$$

or

$$(a+b+c)(bc+ca+ab)=abc(8\tau^2+1).$$

That is

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=8\tau^2+1.$$

Consequently the triangle PQR is self-conjugate with respect to the second circle, provided that

$$8\tau^2=9\sigma^2-1,$$

which will be seen to agree with Prof. Smart's result.

I do not think that the present method gives any useful results in the second part of the proof.

T. J. I'A. B.

GLEANINGS FAR AND NEAR.

622. Norwood's measurement of the arc of the meridian between London and York which took place in 1633, gave results which have since been shown to be correct, and were doubtless known to Newton. But his measurement differed too considerably from those which preceded it to be admitted on the strength of the imperfect apparatus employed by him. Norwood's measure is called by Delambre a great piece of good fortune.—De Morgan, *Picard*, P.C. 18, p. 145. "Sometimes I measured, sometimes I paced, and I believe I am within a scantling of the truth."—Norwood, P.C. 16, p. 330.

623. W. G. Ward had the extreme innocence of a child and a mathematician.—Lytton Strachey, *Eminent Victorians*, p. 33c.

624. Marshal Saxe, aet. 31, left fragments of autobiography in MSS.: I remember that my teachers themselves proposed to have an iron machine put on me to compress my skull, asserting that it was half open. I learnt much quickly, as the military exercise and mathematics; they were obliged to give up reading; for when I studied in a book, and I was asked where I was, and what I had read, I did not know a syllable; it was no better with arithmetic if I was required to do sums on paper, but when I was allowed to calculate in my head, there were no sums which I had not worked sooner than others could work them with pen and ink. I was exactly like the Devil, who does what he is not asked to do; and I learnt perfect Dutch in less than six months without a teacher. . . .

625. To test one people by another is to argue within a very small segment of a circle.—HARRIET MARTINEAU, *Morals and Manners*, p. 14.

626. *Bosola.* Didst thou never study the mathematics?

Old Lady. What's that, sir?

Bosola. Why, to know the trick how to make a many lines meet in one centre.—Webster: *Duchess of Malfi*, Act II. Sc. ii. (Per Mr. T. Puryer White.)

627. Pope Sylvester II. wrote a treatise on the Sphere, on Arithmetic and Geometry, published some years since in Paris.—Goldsmith, *On Polite Learning*. [Gebert died in 1003.]

NOTE ON TANGENTIAL EQUATIONS OF CONICS.

By C. Fox, M.A.

§ 1. If the equation of a conic is

$$ax^2+2hxy+by^2+2gx+2fy+c=0, \tag{1.1}$$

then its tangential equation * is

$$Al^2+2Hlm+Bm^2+2Gl+2Fm+C=0, \tag{1.2}$$

where A, H, B, etc., are respectively the minors of a, h, b, etc., in the determinant

$$\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

Now it is well known that the discriminant of (1.2) is Δ^2, and so if $\Delta=0$ the left-hand sides of both (1.1) and (1.2) factorize. But the interpretation of $\Delta=0$ for (1.1) is that the conic degenerates into two straight lines, whereas the interpretation for (1.2) is that the conic degenerates into two points. This apparent discrepancy is explained by the following result.

If (1.1) *breaks up into two different factors, then* (1.2) *is a perfect square and is the equation of the point of intersection of the lines whose equation is given by* (1.1).

One method of proof is to find the tangential equation of

$$(a_1x+b_1y-1)(a_2x+b_2y-1)=0. \tag{1.3}$$

The condition that (1.3) should be touched by the line $lx+my=1$ turns out to be

$$[(b_1-b_2)l-(a_1-a_2)m+(a_1b_2-a_2b_1)]^2=0, \tag{1.4}$$

which is evidently the equation of the point

$$\left(\frac{b_1-b_2}{a_1b_2-a_2b_1}, \quad -\frac{a_1-a_2}{a_1b_2-a_2b_1}\right),$$

and this is the point of intersection of the two lines of (1.3).

Another method of proof is to note that if $\Delta=0$, then the first minors of

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

[the discriminant of (1.2)] all vanish. For

$$AB-H^2=\Delta c=0,$$

and similarly for the other first minors.

This is a necessary and sufficient condition that the left-hand side of (1.2) should be a perfect square.

The converse of the above theorem is as follows.

If (1.2) *breaks up into two factors representing two different points, then* (1.1) *is a complete square, and is the equation of the line joining these points.*

This is easily proved by the same methods.

§ 2. In this paragraph I deal with the following problem. If we choose λ so that the left-hand side of

$$Al^2+2Hlm+Bm^2+2Gl+2Fm+C+\lambda=0 \tag{2.1}$$

breaks up into two factors representing two points, what is the geometrical relationship between these points and the conic (2.1) ? †

* See *e.g.* Smith's *Conic Sections*, Chapter XII.

† This is the tangential analogue of finding the asymptotes of a given conic.

The answer is as follows:

We can choose λ in one and only one way so that (2.1) *degenerates into two points, and when λ is so chosen then these two points are the points of contact of the tangents from the origin to the conic* (2.1).

To prove this, let $S_1=0$ and $S_2=0$ be the tangential equations of two different conics, then for different values of λ

$$S_1+\lambda S_2=0$$

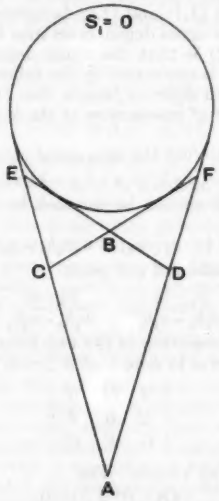
gives us the family of conics which touch the four common tangents of $S_1=0$ and $S_2=0$.

For three values of λ the left-hand side of (2.1) breaks up into two factors, that is to say, for three values of λ (2.1) degenerates into two points. Let

$$S_1 \equiv Al^2+2Hlm+Bm^2+2Gl+2Fm+C$$

and

$$S_2 \equiv (x_1l+y_1m-1)\,(x_2l+y_2m-1).$$



In the diagram (x_1, y_1) are the Cartesian co-ordinates of the point A and (x_2, y_2) those of the point B. Evidently

$$S_1+\lambda S_2=0 \qquad (2.2)$$

represents a family of conics touching the four tangents from A and B to $S_1=0$, and the three pairs of points which are the limiting cases of members of the family are (i) A and B, (ii) C and D, (iii) E and F. The value of λ which makes (2.2) degenerate into A and B is infinite.

Now let B tend to A, *i.e.* $x_2 \to x_1$ and $y_2 \to y_1$.

Then C and D also tend to A, and so the value of λ which makes (2.2) degenerate into C and D must tend to infinity. The remaining value of λ which makes (2.2) degenerate into E and F tends to a finite limit, whilst E and F tend to the points of contact of the tangents from A to $S_1=0$.

It follows that there is only one finite value of λ which makes

$$S_1+\lambda\,(x_1l+y_1m-1)^2=0 \qquad (2.3)$$

degenerate into two points, and that these two points are the points of contact of the tangents from A to $S_1=0$. In the particular case when $x_1=y_1=0$, that is, when A is at the origin, this becomes the solution of the problem stated at the beginning of the paragraph.

It is worth noting that for varying λ (2.3) is the equation of the family of conics touching two fixed lines at fixed points, and that (x_1, y_1) are the Cartesian co-ordinates of the point of intersection of these lines. C. Fox.

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628. [In this, the first enunciation of Taylor's Theorem, we find the symbols $\therefore$ and $\because$ used as follows:]

$$x+\dot{x}\frac{v}{1\dot{z}}+\ddot{x}\frac{\dot{v}v}{1.2\dot{z}^2}+\dddot{x}\frac{\dot{v}\dot{v}v}{1.2.3\dot{z}^3}+\dots, \text{ etc.},$$

and

$$x+\dot{x}\frac{v}{1\dot{z}}+\ddot{x}\frac{v^2}{1.2\dot{z}^2}\therefore\dddot{x}\frac{v^3}{1.2.3\dot{z}^3}+\dots, \text{ etc.}$$

Brook Taylor, *Methodus Incrementorum*, Prop. VII. Theor. III. and Coroll. II.

Of this volume Bernoulli wrote (Aug. 1716): "I have at length received Taylor's book. What, in the name of God, does the man mean by the darkness in which he involves the clearest things! No doubt to conceal his habit of thieving; as far as I can make it out, I see nothing but what he has stolen from me, through his thick cloud of obscurity. . . ."—*v.* De Morgan, Brook Taylor, P.C. 24, p. 126. [The use lasted over a century.—v. *Quarterly Review*, May 1811, p. 348.]

629. Early writers divided Arithmetic into digital, articulate and compound digital, the first nine, counted on the fingers; articulate, multiples of ten and compound all others . . . made up by jointly adding digital and articulate. The Arabic word for *articulate* meant *knots*.

630. The lithe and sinewy Wilson—smiling—once a Senior Wrangler, says some one—who conjures for us with sine and secant and plays all sorts of pretty tricks with angles.—C. Mosley, *Cornhill*, Feb. 1915, [on Canon J. M. Wilson, the *doyen* of the Mathematical Association].

631. Southey to Coleridge. . . . Davies Giddy, whose face ought to be perpetuated in marble, for the honour of maths. Such a forehead I never saw. —*Athenaeum*, 1850 [later was Davies Gilbert, Treasurer Royal Soc.].

632. Earl Stanhope has at last lifted his eyes from Euclid, and directed them to matrimony.—[The second Earl]. Walpole to Mann, March 4th, 1745.

633. One morning Herschel and I called on Laplace, who spoke to us of various English works on mathematical subjects. Amongst others, he mentioned with approbation, "Un ouvrage de vous deux." We were both quite at a loss to know to what work he referred.* Herschel and I had not written any joint work, although we had together translated the work of Lacroix. The volume of the *Memoirs of the Analytical Society*, though really our joint production, was not known to be such, and it was also clear that Laplace did not refer to that production. Perceiving that we did not recognise the name of the author to which he referred, Laplace varied the pronunciation by calling him *vous deux*; the first word being pronounced as the French word "vous" and the second as the English word "deuce."

Upon further explanation it turned out that Laplace meant to speak of a work published by Woodhouse, whose name is in the pronunciation of the French so very like *vous deux*.—C. Babbage, *Passages from the Life of a Philosopher*, 1864.

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* ? R. Woodhouse, *Treatise on Astronomy*, 1812-1818, or *Treatise on Isoperimetrical Problems, and the Calculus of Variations*, 1810.

HOMOGENEOUS LINE COORDINATES.

By M. A. Porter, M.A.

1. *Coordinates.*

p, q, r are lengths of perpendiculars from A, B, C on line, and have same or opposite signs according as A, B, C are on same or opposite sides of the line.

The coordinates given by $x=ap$, $y=bq$, $z=cr$ are more convenient.

Identity.

With usual notation the coordinates (x, y, z) of a line are connected by the relation $\Sigma x^2 - 2\Sigma yz \cos A = 4\Delta^2$.*

If θ is the angle which (x, y, z) makes with BC,

$$a \sin\theta = q - r, \quad c\sin(B+\theta) = p - q,$$

and the above identity is obtained by eliminating θ.

2. Any line is given by coordinates (x, y, z) or by ratios of coordinates $(x : y : z)$, the actual coordinates being obtained by means of the above identity.

3. *Angle between two lines.*

The angle between two lines $(x_1, y_1, z_1)(x_2, y_2, z_2)$ is given by

$$\cos\phi_{12} = [\Sigma x_1x_2 - \Sigma(y_1z_2 + y_2z_1)\cos A]/4\Delta^2.$$

Corollaries. (x, y, z) makes with sides of triangle angles given by

$$\cos\theta = (x - z\cos B - y\cos C)/2\Delta, \text{ etc.}$$

We can write $\cos\phi_{12} = (x_1\cos\theta_2 + y_1\cos\phi_2 + z_1\cos\psi_2)/2\Delta$

$$= (x_2\cos\theta_1 + y_2\cos\phi_1 + z_2\cos\psi_1/2\Delta.$$

4. *Condition for concurrency of three lines.*

$(x_1, y_1, z_1)(x_2, y_2, z_2)(x_3, y_3, z_3)$ are concurrent if $\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} = 0.$

4*a*. Special lines connected with triangle of reference.

Medians : $(0 : b : -c)$, $(-a : 0 : c)$, $(a : -b : 0)$.

Altitudes :

$(0 : \cos B : -\cos C)$, $(-\cos A : 0 : \cos C)$, $(\cos A : -\cos B : 0)$.

Internal bisectors of angles : $(0 : 1 : -1)$, $(-1 : 0 : 1)$, $(1 : -1 : 0)$.

External bisectors of angles : $(0 : 1 : 1)$, $(1 : 0 : 1)$, $(1 : 1 : 0)$.

Perpendicular bisectors of sides :

$$(b^2 - c^2 : ab : -ac)\,(-ab : c^2 - a^2 : bc)\,(ac : -bc : a^2 - b^2).$$

These all satisfy condition for concurrency.

5. *Equation of a point.* $lx + my + nz = 0$.

Any point on line joining $\Sigma l_1x = 0$, $\Sigma l_2x = 0$ is $\Sigma(l_1 + \lambda l_2)x = 0$.

Any line through point of intersection of (x_1, y_1, z_1), (x_2, y_2, z_2) is

$$(x_1 + \lambda x_2,\ y_1 + \lambda y_2,\ z_1 + \lambda z_2).$$

6. *Bisectors of angles.*

The bisectors of angles between the two lines

$$(x_1, y_1, z_1), \quad (x_2, y_2, z_2)$$

are $(x_1 - x_2 : y_1 - y_2 : z_1 - z_2)$ and $(x_1 + x_2 : y_1 + y_2 : z_1 + z_2)$.

[To distinguish between bisectors : If a majority of positive signs are given always to the coordinates of a line *i.e.* $++-$, or $+++$, then the line

* Cf. C. A. Scott, *Modern Analytical Geometry*, p. 16.

$(x_1-x_2, y_1-y_2, z_1-z_2)$ will be the bisector of the angle containing the majority of the vertices of the triangle of reference.

N.B.—This breaks down in certain cases when a vertex lies on one or both of the lines.]

7. *Length of perpendicular.*

The length of the perpendicular from the point $\Sigma lx=0$ on the line (x_1, y_1, z_1) is $\Sigma lx_1/\Sigma la$, and, according as this is $+^{ve}$ or $-^{ve}$, $\Sigma lx=0$ lies on the $+^{ve}$ or $-^{ve}$ side of the line (*i.e.* side with majority or minority of vertices).

If p is the length of perpendicular, then the line through $\Sigma lx=0$ parallel to (x_1, y_1, z_1) is (x_1-ap, y_1-bp, z_1-cp). Substituting in $\Sigma lx=0$ we get above formula.

8. Line joining $\Sigma l_1x=0$, $\Sigma l_2x=0$ is

$$(m_1n_2-m_2n_1 : n_1l_2-n_2l_1 : l_1m_2-l_2m_1).$$

Point of intersection of (x_1, y_1, z_1), (x_2, y_2, z_2) is $\Sigma x(y_1z_2-y_2z_1)=0$.

9. Special points connected with triangle of reference.

Mid-points of sides : $y/b+z/c=0$, etc.

Median centre : $\Sigma x/a=0$.

In-centre : $\Sigma x=0$.

Ex-centres : $-x+y+z=0$, $x-y+z=0$, $x+y-z=0$.

Orthocentre : $\Sigma x/\cos A=0$.

Circumcentre : $\Sigma x\cos A=0$.

10. *Special lines.*

The line joining circum- and orthocentres is

$$(\sin 2A\sin(B-C), \sin 2B\sin(C-A), \sin 2C\sin(A-B)).$$

This satisfies $\Sigma x/a=0$; $\therefore$ median centre lies on line joining circum- and ortho-centres.

The line joining A to point on BC dividing it in the ratio $\lambda_1 : \mu_1$ is

$$(0 : b\lambda_1 : -c\mu_1).$$

The three lines $(0 : b\lambda_1 : -c\mu_1)$, $(-a\mu_2 : 0 : c\lambda_2)$, $(a\lambda_3 : -b\mu_3 : 0)$ are concurrent if $\lambda_1\lambda_2\lambda_3=\mu_1\mu_2\mu_3$. [Ceva's Theorem.]

The three points $y/(b\lambda_1)+z/(c\mu_1)=0$, etc., are collinear if $\lambda_1\lambda_2\lambda_3=-\mu_1\mu_2\mu_3$. [Menelaus' Theorem.]

The line through intersection of (x_1, y_1, z_1), (x_2, y_2, z_2) perpendicular to (x_3, y_3, z_3) is $(E_{23}x_1-E_{31}x_2 : E_{32}y_1-E_{31}y_2 : E_{32}z_1-E_{31}z_2)$

where $E_{23}=\Sigma x_2x_3-\Sigma(y_2z_3+y_3z_2)\cos A=4\Delta^2\cos\phi_{23}$.

From this we easily deduce the concurrence of the altitudes of the triangle

$$(x_1, y_1, z_1), \quad (x_2, y_2, z_2), \quad (x_3, y_3, z_3)$$

11. *Circle.*

The circle centre $\Sigma l_1x=0$, radius r is

$$(\Sigma l_1x)^2=r^2(\Sigma l_1a)^2E(xyz)/(4\Delta^2)$$

where $$E\equiv\Sigma x^2-2\Sigma yz\cos A.$$

In-circle of Δ of reference is $\Sigma yz(1+\cos A)=0$.

Ex-circles are $yz(1+\cos A)+zx(-1+\cos B)+xy(-1+\cos C)=0$, etc.

The fourth common tangent of in-circle and ex-circle is

$$(1+\cos A : \cos C-\cos B : \cos B-\cos C);$$

this passes through $y+z=0$, the point where the internal bisector of $\hat{A}$ meets BC.

12. *Conics.*

The line at infinity is $(a : b : c)$.

$\Sigma lx=0$ is a point at infinity if $\Sigma la=0$.

Conics defined by property $SY \cdot S'Y'=b^2$.

$$\phi(xyz)\equiv(\Sigma l_1x\Sigma l_2x)-E(xyz)=0$$

is a conic with foci $\Sigma l_1x=0$, $\Sigma l_2x=0$;

this is a parabola if $\Sigma l_1a\Sigma l_2a=0$,

i.e. if $\phi(abc)=0$ since $E(abc)\equiv 0$,

showing that a parabola, defined as a conic having one focus at infinity, touches the line at infinity.

The conic is an ellipse or hyperbola according as

$$\phi(abc) \text{ is } +^{ve} \text{ or } -^{ve}.$$

N.B.—It is very difficult if not impossible to bring the general quadratic to above form.

For a circle $l_1, m_1, n_1=l_2, m_2, n_2$.

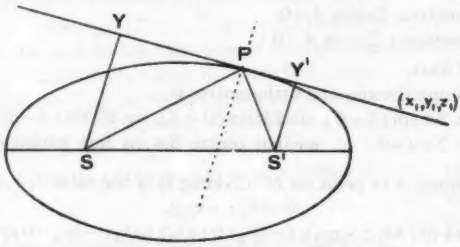
For a conic touching sides of triangle of reference

$$l_1l_2=m_1m_2=n_1n_2 \text{ [circles given by } l_1^2=m_1^2=n_1^2\text{]},$$

and equation reduces to $\Sigma Ayz=0$.

Point of contact of tangent (x_1, y_1, z_1) is

$$\Sigma l_1x\Sigma l_2x_1+\Sigma l_1x_1\Sigma l_2x-(\Sigma xx_1-\Sigma(y_1z+yz_1)\cos A)=0.$$



Let foci S, S' be $\Sigma l_1x=0$, $\Sigma l_2x=0$ respectively.

Let (x', y', z') be the line through P perpendicular to (x_1, y_1, z_1), then

$$\Sigma l_1x'\Sigma l_2x_1+\Sigma l_1x_1\Sigma l_2x'=0,$$

$$\text{i.e. } YP \cdot S'Y'-SY \cdot Y'P=0,$$

$$\text{i.e. } SY : YP=S'Y' : Y'P,$$

$$\text{i.e. } S\hat{P}Y=S'\hat{P}Y'.$$

Confocal conics. $\Sigma l_1x\Sigma l_2x-\lambda_1E(xyz)=0$,

$\Sigma l_1x\Sigma l_2x-\lambda_2E(xyz)=0$.

Solving these equations we get common tangents; they are given by $\Sigma l_1x\Sigma l_2x=0$, *i.e.* four common tangents are tangents from the foci.

13. *Note on non-homogeneous line coordinates.*

x, y are perpendiculars on line from points XY.

$ax+by=0$ is point P on XY dividing it in the ratio $b : a$.

Perpendicular from $ax+by=0$ on (x_1, y_1), is $(ax_1+by_1)/(a+b)$.

Conic foci $a_1x+b_1y=0$, $a_2x+b_2y=0$ is $(a_1x+b_1y)(a_2x+b_2y)=c$.

[Cf. point equation of conic, factors giving asymptotes.]

Point at infinity on XY is $x-y=0$;

$$\therefore \text{ parabola is } (ax+by)(x-y)=c.$$

Equations of points not on XY and conics not symmetrical about XY are not linear and quadratic equations, respectively.

14. *Force resolved into components along sides of triangle of reference.*

A force F acting along line (x, y, z) is equivalent to forces $Fx/2\Delta$, etc., along BC, etc.

Forces P, Q, R along sides of triangle have a resultant F acting along the line $(P : Q : R)$, the magnitude of F being given by $F^2 = \Sigma P^2 - 2\Sigma QR \cos A$.

[*N B.*—$R=0$ gives ordinary formula.]

If $P : Q : R = a : b : c$, forces are equivalent to a couple. M. A. PORTER.

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634. P.S.—Send me a good Cocker, or the best simplifier of arithmetic. I cast up my household accounts and settle them daily myself, and you cannot imagine the difference.—Byron to Kinnaird, 10^{bre} 19th, 1822.

635. A proper edition of Newton's philosophical works, prepared with that learning and ability which became the author and his subject, is at this time a great desideratum. . . . The public has a right to look for a good edition . . . to that university, which boasts his illustrious name, and where his works are peculiarly studied. Ought it not publicly to authorize some of its ablest mathematicians to prepare such an edition? . . .—*Quarterly Review*, 1810, p. 399.

636. He was not distinguished from other men, by any singularity either natural or affected.—Fontenelle of Sir Isaac Newton.

637. A dictionary of arithmetic or geometry can serve only to confound. —Johnson's Preface to Rolt's *New Dict. of Trade and Commerce*, 1757.

638. He was preparing to dive for treasure in a new-invented bell; and another was on the point of discovering the longitude.—Dr. Johnson, *The Rambler*, No. 67.

Type of the fall of Greece and Rome
While more than mathematic gloom
Envelops all around.

On an Eagle confined in a College Court. Christopher Smart.

639. Tout à coup, saisissant une vieille géométrie ornée d'assez méchantes figures de Sébastien Leclerc : *

" Peut-être, reprit-il, au lieu de me noyer dans l'amour ou dans l'eau, si je n'étais chrétien et catholique, prendrais-je le parti de me jeter dans la mathématique, où l'esprit trouve les aliments dont il est le plus avide, à savoir : la suite et la continuité. Et j'avoue que ce petit livre, tout commun qu'il est, me donne quelque estime du génie de l'homme."

À ces mots, il ouvrit si largement le traité de Sébastien Leclerc, à l'endroit des triangles, qu'il faillit le rompre net. Mais bientôt il le rejeta avec dégoût.

" Hélas ! murmura-t-il, les nombres dépendent du temps, les lignes, de l'espace, et ce sont là encore des illusions humaines. En dehors de l'homme, il n'y a ni mathématique, ni géométrie, et c'est en définitive une connaissance qui ne nous fait pas sortir de nous-mêmes, bien qu'elle affecte un air d'indépendance assez magnifique."

Note de l'éditeur.—La géométrie dont parle Jacques Tournebroche est ornée de figures de Sébastien Leclerc dont j'admire au contraire la précise élégance et la fine exactitude. Mais il faut souffrir la contradiction.—Anatole France, *Les Opinions de M. Jérôme Coignard*, cap. ix. [per Mr. F. Puryer White].

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* The book referred to is *Pratique de la Géométrie sur le papier et sur le terrain*, par Sebastien Leclerc, Paris, 1669. It is the subject of an article by J. H. Graf in *Abhandlungen zur Geschichte der Mathematik*, Heft 9, 1899, pp. 115-22. There were many later editions Paris, 1682, 1700, 1716, 1735, 1744; translated into Latin, Amsterdam, 1692; into English, under the title *Magnum in Parvo: or, the Practice of Geometry*, by Robert Fricke, London, 1672 (no mention of Leclerc), later editions, with attribution to Leclerc, 1727 1742; Italian translation, Rome 1761. [Per F. Puryer White.]

THE TETRAHEDRON AND A TWELVE-POINTS SPHERE.

By R. T. Robinson, M.A.

In a tetrahedron the sphere through the c.g.'s of the four faces passes also through two other sets of four points.

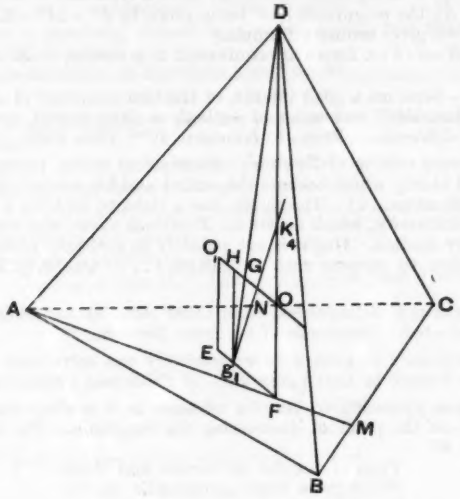


Fig. 1.

Case I. If the opposite edges of the tetrahedron $ABCD$ are mutually perpendicular, the $\perp^{rs}$ from A, B, C, D to the opposite faces intersect in a point O, and the sphere through the c.a.'s of the faces passes through

(1) the feet of the $\perp^{rs}$ from A, B, C, D to the opposite faces;

(2) the points $K_1K_2K_3K_4$ where $OK_1=\frac{1}{3}OA$, $OK_2=\frac{1}{3}OB$, etc.

Let O_1 be the centre of the sphere $ABCD$,

G be the c.g. of the tetrahedron $ABCD$,

g_1 be the c.g. of the face ABC,

E and F be the feet of the $\perp^{rs}$ from O_1 and D respectively to the face ABC.

Let O_1G meet DF in O, and draw g_1H parallel to DF to meet O_1G in H.

1. F is the ortho-centre of $\triangle ABC$, E is the centre of the $\odot ABC$;

$$\therefore\ Eg_1=\tfrac{1}{2}\,.\,g_1F \text{ and } O_1H=\tfrac{1}{3}O_1O;$$
$$\therefore\ O_1G=\tfrac{1}{3}OO_1+HG=\tfrac{1}{3}OO_1+\tfrac{1}{3}OG;$$
$$\therefore\ OG=O_1G=\tfrac{1}{3}OO_1;$$
$$\therefore\ DF \text{ meets } O_1G \text{ in a fixed point.}$$

Similarly the $\perp^{rs}$ from ABC to the opposite faces meet O_1G in the same fixed point O;

$\therefore$ O is the intersection of the $\perp^{rs}$ from the vertices to the opposite faces.

2. On O_1G produced take a point N so that $NG=\frac{1}{2}O_1G$,

then $NG=\frac{1}{2}OG=HG$;

$\therefore\ HN=HG+GN=\frac{3}{2}OG=\frac{1}{2}O_1O.$

Since $O_1H=\frac{1}{2}O_1O,\ NO=\frac{1}{2}O_1O$;

$\therefore\ HN=NO$;

$\therefore\ Ng_1=NF.$

3. $O_1G:GN=3:1$ and $O_1O:ON=3:1$;

$\therefore$ G and O are the centres of similitude of spheres centres O_1 and N whose radii are in the ratio of 3 to 1.

4. O_1 being the centre of the sphere $ABCD$, N is evidently the centre of the sphere through $g_1g_2g_3g_4$, the C.G.'s of the faces ABC, etc., because G is a centre of similitude for the two figures $ABCD$, $g_1g_2g_3g_4$, and $O_1G=3\,.\,GN$.

5. From 2, $Ng_1=NF$; $\therefore$ the sphere $g_1g_2g_3g_4$ passes through F the foot of the $\perp^r$ from D to the face ABC. Similarly it passes through the feet of the $\perp^{rs}$ from ABC to the opposite faces.

6. Since O is an external centre of similitude if $OK_4=\frac{1}{2}OD$, the sphere $g_1g_2g_3g_4$ centre N passes through K_4. Similarly it passes through $K_1K_2K_3$.

7. If ρ is the radius of the self-polar sphere of the tetrahedron $ABCD$, $\rho^2=-OD\,.\,OF$, and O is the centre of this sphere;

$\therefore$ D and F are inverse points w.r.t. the self-polar sphere.

Similarly A and the foot of the $\perp^r$ from A to BCD,
B „ „ „ „ „ B „ ACD,
C „ „ „ „ „ C „ ABD,
are inverse points w.r.t. this sphere;

$\therefore$ the sphere $g_1g_2g_3g_4$ is the inverse of the sphere $ABCD$ w.r.t. the self-polar sphere.

8. If DO meets the sphere $ABCD$ again in X, then since O is a centre of similitude and F is on the sphere $g_1g_2g_3g_4$, $OF=\frac{1}{3}OX$.

9. If ρ_1 is the radius of the sphere $ABCD$,

$$\rho_1^2-\rho^2=\frac{\lambda^2}{4},$$

where $$BC^2+AD^2=AC^2+BD^2=AB^2+CD^2=\lambda^2,$$
$$i.e.\quad a^2+d^2=b^2+e^2=c^2+f^2=\lambda^2.$$

$$Hg_1=\tfrac{1}{3}DO,\ \text{but}\ Hg_1=\frac{2O_1E+OF}{3};$$

$$\therefore\ DO=2O_1E+OF.$$

$$\rho_1^2-\rho^2=R^2+O_1E^2+OD\,.\,OF \text{ where } R \text{ is radius of } \odot ABC$$
$$=R^2+\tfrac{1}{4}(DO-OF)^2+OD\,.\,OF=R^2+\tfrac{1}{4}(DO+OF)^2$$
$$=R^2+\frac{DF^2}{4}=R^2+\tfrac{1}{4}(d^2-AF^2)$$
$$=R^2+\tfrac{1}{4}(a^2+d^2-a^2-4R^2\cos^2 BAC)$$
$$=\frac{\lambda^2}{4}+R^2\sin^2 BAC-\frac{a^2}{4}=\frac{\lambda^2}{4}.$$

Case II. Any tetrahedron $ABCD$.

Let O_1 be centre of sphere $ABCD$;
F be foot of $\perp^r$ from D to the plane ABC;
Ω „ „ „ O_1 „ „ „
i.e. Ω is centre of the $\odot ABC$;

K be the ortho-centre of the $\triangle ABC$;
G „ c.g. of tetrahedron $ABCD$;
g_1 „ „ the face ABC;
M be the foot of the $\perp^r$ from G to ABC.

To avoid confusion the points ABC have not been inserted in the figure. The plane ΩKF is the plane ABC.

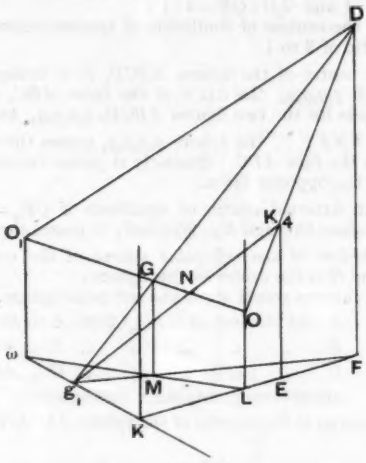


FIG. 2.

1. Let the plane ΩO_1GM meet the plane DFK in OL,
then $\Omega g_1 = \frac{1}{4}\Omega K$ and $g_1M = \frac{1}{4}g_1N$.

ΩML is a transversal for the $\triangle g_1FK$;

$$\therefore \left(\frac{K\Omega}{\Omega g_1}\cdot\frac{g_1M}{MF}\cdot\frac{FL}{LK}\right) = -1 \quad \text{or} \quad 3\times\tfrac{1}{3}\frac{FL}{LK} = 1,$$

$$\text{or} \quad FL = LK;$$

$\therefore$ L is the mid-point of FK.

g_1MF is a transversal for the $\triangle \Omega KL$;

$$\therefore \left(\frac{KF}{FL}\cdot\frac{LM}{M\Omega}\cdot\frac{\Omega g_1}{g_1K}\right) = -1 \quad \text{or} \quad 2\times\frac{LM}{M\Omega}\times\tfrac{1}{2} = 1,$$

$$\text{or} \quad LM = M\Omega;$$

$$\therefore \quad OG = O_1G.$$

2. Take a point N in O_1G produced so that $NG = \frac{1}{3}O_1G$,
then, as before, $O_1G : GN = OO_1 : ON = 3 : 1$;

$\therefore$ G and O are the centres of similitude for the spheres $ABCD$, $g_1g_2g_3g_4$, whose centres are O_1 and N respectively.

3. If g_1N is produced to meet OD in K_4, then since $NG = \frac{1}{3}O_1G$ and $Gg_1 = \frac{1}{3}GD$, Ng_1 is parallel to O_1D and equal to $\frac{1}{3}O_1D$, and

$$NK_4 : O_1D = ON : OO_1 = 1 : 3;$$

$$\therefore \quad NK_4 = \tfrac{1}{3}O_1D;$$

$\therefore$ the sphere $g_1g_2g_3g_4$ passes through K_4. Similarly it passes through $K_1K_2K_3$ where $OK_1=\frac{1}{3}OA$, $OK_2=\frac{1}{3}OB$, etc.

4. $Ng_1=NK_4=NE$ where E is the foot of the $\perp^r$ from K_4 to the plane ABC;

$\therefore$ the sphere $g_1g_2g_3g_4$ passes through the foot of the $\perp^r$ from K_4 to the plane ABC. Similarly it passes through the feet of the $\perp^{rs}$ from K_1 to BCD, K_2 to ACD, K_3 to ABD.

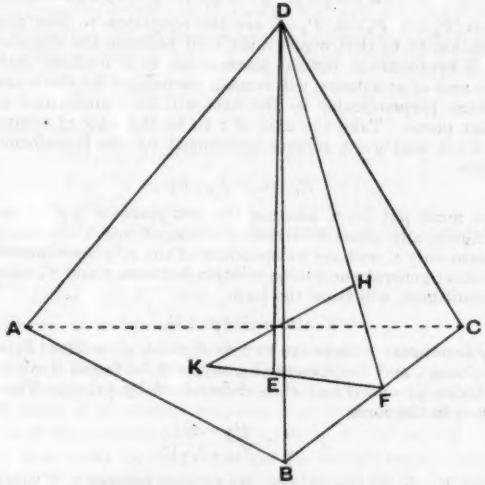


FIG. 3.

5. The point O is the centre of the equilateral hyperboloid through $ABCD$.

If AG is produced to A_1 so that $GA_1=GA$,

BG „ „ B_1 „ $GB_1=GB$,

and similarly for CG and DG, we get another tetrahedron $A_1B_1C_1D_1$ which is such that

B_1C_1 is parallel to BC and AD and B_1C_1 bisect each other,

C_1A_1 „ „ CA „ BD and C_1A_1 „ „

and so on.

Since $O_1G=GO$, O is the centre of the sphere $A_1B_1C_1D_1$.

6. If st. lines be drawn through the ortho-centres of the faces DBC, DCA, DAB $\perp^r$ to these faces to meet the base ABC in KLM respectively, then A, B, C, K, L, M and E the foot of the $\perp^r$ from D to ABC all lie on a rectangular hyperbola.

Let H be the ortho-centre of ABC, and let HK be $\perp^r$ to the plane DBC, then

$$BF \cdot FC = FH \cdot FD$$
$$= FE \cdot FK,$$

and BC is $\perp^r$ to KF for it is $\perp^r$ to DF and HK;

$\therefore$ K lies on the rectangular hyperbola through ABC and E.

Similarly L and M lie on this $R.H.$

R. T. ROBINSON.

MATHEMATICAL NOTES.

913. [T. 3. a.] *Note on Geometrical Optics.**

§1. Geometrical Considerations.

Any one-to-one transformation (in which planes correspond to planes) is given by formulae of the type

$$x'=P_1/P_4, \quad y'=P_2/P_4, \quad z'=P_3/P_4, \ldots\ldots (1)$$

where $P_1=0$, $P_2=0$, $P_3=0$, $P_4=0$ are the equations to four planes. If we take the relation to be that which must hold between the objects and images formed by a symmetrical optical instrument, it is evident that any plane through the axis of symmetry will remain unchanged by the transformation; and any plane perpendicular to the axis will be transformed into another perpendicular plane. Take the axis of z to be the axis of symmetry; then the planes $x=0$, and $y=0$, remain unchanged by the transformation; thus we must have

$$P_1=ax, \quad P_2=by.$$

Further, we must put $b=a$, because the two planes $x'\pm y'=0$ correspond to $x\pm y=0$. Again, any plane $z=$const. is changed into $z'=$const.; and so P_3 and P_4 contain only z, and are independent of the x, y coordinates.

Thus the most general one-to-one relation between z and z', consistent with our other conditions, will be of the form

$$z'=(\alpha z+\beta)/(\gamma z+\delta) \ldots\ldots (2)$$

Now, if γ is not zero,† there are two *focal points*; the first (F_1, say) is given by making $z'\to\infty$, and the second (F_2, say) will be found if we make $z\to\infty$. Thus F_1 satisfies $\gamma z+\delta=0$ and F_2 is determined by $\gamma z'=\alpha$. The relation (2) can be written in the form

$$z'-\frac{\alpha}{\gamma}=\frac{\beta\gamma-\alpha\delta}{\gamma^2(z+\delta/\gamma)};$$

and so, taking F_1, F_2 as new origins, the relation between z, z' takes the simple form

$$z'=c/z. \ldots\ldots (3)$$

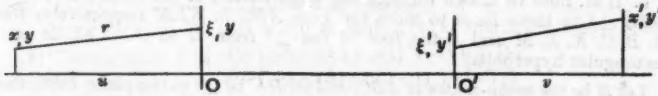
Using (3) in (1) we have now the formulae (due to Newton):

$$x'=\frac{ax}{z}, \quad y'=\frac{ay}{z}, \quad z'=\frac{c}{z}, \ldots\ldots (4)$$

but we cannot deduce any relation between the coefficients a, c without introducing some optical considerations as in §2 below.

§2. Optical Considerations.

We can obtain all that is needed from the method of the *reduced path*: but to economize symbols, we shall assume that the special instrument has its first and last refractive indices each equal to unity.‡



* v. Review, p. 309.

† When γ is zero, P_4 is a constant and the system is *telescopic*, the formulae being of the type

$$x'=kx, \quad y'=ky, \quad z'=\lambda z+\theta.$$

The optical relation found as in §2 is then $\lambda=k^2\mu'/\mu$.

‡ Thus we exclude immersion-objectives for microscopes, but we include any ordinary microscope, telescope or photographic lens.

Then if we fix our attention on two points $(\xi, \eta, 0)$, $(\xi', \eta', 0)$ on the planes of reference (which at present are any two convenient planes perpendicular to the axis), the reduced path between them will be of the form

$$\phi(X, Y, Z)$$

where $$X=\xi^2+\eta^2, \quad Y=\xi\xi'+\eta\eta', \quad Z=\xi'^2+\eta'^2 \qquad \ldots\ldots(5)$$

Thus the complete reduced path from $(x, y, -u)$ to (x', y', v), *via* the points selected will be

$$V=\sqrt{u^2+(\xi-x)^2+(\eta-y)^2}+\phi(X, Y, Z)+\sqrt{v^2+(\xi'-x')^2+(\eta'-y')^2} \qquad \ldots\ldots 6)$$

The optical properties are expressed by the conditions that V should be stationary for displacement of ξ, η, ξ', η'; thus we must have the four equations:

$$\frac{\partial V}{\partial \xi}=0, \quad \frac{\partial V}{\partial \eta}=0, \quad \frac{\partial V}{\partial \xi'}=0, \quad \frac{\partial V}{\partial \eta'}=0 \; ; \qquad \ldots\ldots(7)$$

and these give the results:

$$\left.\begin{aligned} l+2\xi\phi_1+\xi'\phi_2=0, \qquad & m+2\eta\phi_1+\eta'\phi_2=0, \\ -l'+\xi\phi_2+2\xi'\phi_3=0, \qquad & -m'+\eta\phi_2+2\eta'\phi_3=0, \end{aligned}\right\} \qquad \ldots\ldots(8)$$

where $$\phi_1=\partial\phi/\partial X, \quad \phi_2=\partial\phi/\partial Y, \quad \phi_3=\partial\phi/\partial Z,$$

while (l, m, n), (l', m', n') are the direction-cosines of the initial and final positions of the ray, so that

$$\left.\begin{aligned} l=(\xi-x)/r, \qquad & m=(\eta-y)/r, \qquad n=u/r, \\ l'=(x'-\xi')/r', \quad & m'=(y'-\eta')/r', \quad n'=v/r', \end{aligned}\right\} \qquad \ldots\ldots(9)$$

with $$r^2=u^2+(\xi-x)^2+(\eta-y)^2, \quad r'^2=v^2+(x'-\xi')^2+(y'-\eta')^2.$$

From the form of (8) it is evident that *in general the equations for x' will contain y and m, and those for y' will contain x and l, so that it is impossible to obtain a complete theory unless skew rays are discussed.**

On the other hand, *if we consider only rays close to the axis*, the magnitudes X, Y, Z will be of the second order; and $n=\sqrt{1-l^2-m^2}$, $n'=\sqrt{1-l'^2-m'^2}$ will both differ from unity by magnitudes of the second order. *Accordingly, if we neglect third order magnitudes*, the equations (8) and (9) lead to the simpler forms:

$$\left.\begin{aligned} l+A\xi+B\xi'=0, \qquad & m+A\eta+B\eta'=0, \\ -l'+B\xi+C\xi'=0, \qquad & -m'+B\eta+C\eta'=0, \\ \xi=x+lu, \qquad & \eta=y+mu, \\ \xi'=x'-l'v, \qquad & \eta'=y'-m'v, \end{aligned}\right\} \qquad \ldots\ldots(10)$$

where A, B, C are the constant values of $2\phi_1$, ϕ_2, $2\phi_3$ corresponding to $X=0$, $Y=0$, $Z=0$. Thus we have †

$$\phi=U_0+(\tfrac{1}{2}AX+BY+\tfrac{1}{2}CZ)$$
$$+\text{terms of the second and higher orders in } X, Y, Z.$$

When we consider the form of (10) it is evident that *the formulae for x' are independent of y and m, while those for y' are independent of x and l. Thus* **to this order** *the final formulae remain the same, whether skew rays are used or not.*

It is perhaps worth while, however, to see that the formulae (10) lead to the same type of equations as (4); these formulae (10) can be written

$$\left.\begin{aligned} l(1+Au)-l'(Bv)+(Ax+Bx')=0, \\ l(Bu)-l'(1+Cv)+(Bx+Cx')=0, \end{aligned}\right\} \qquad \ldots\ldots(11)$$

omitting, for brevity, the equations in m, m', y, y' (since they have the same coefficients).

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* For if we restrict our work to rays lying in a plane through the axis, that plane can be taken as the plane of zx; then the incident ray will be such that $y=0$, $m=0$, and thus the terms containing y and m will be missing from the formula found for x'.

† The "*first order aberrations*" are derived from the six terms of the second order in ϕ; and there are *five* independent aberrations (see Dr. Steward's *Tract*, pp. 30-32).

Eliminating l' we see that the equations (11) lead to

$$\{(1+Au)(1+Cv)-B^2uv\}\,l+\{A+(AC-B^2)\,v\}\,x+Bx'=0, \quad \ldots\ldots(12)$$

and the equation (12) will hold for *every* ray (provided that l, m are *small*), if we have

$$\left.\begin{aligned}(1+Au)\,(1+Cv)-B^2uv&=0,\\ \{A+(AC-B^2)\,v\}\,x+Bx'&=0,\end{aligned}\right\} \quad \ldots\ldots(13)$$

Similarly by eliminating l from (11) it is clear that we must also have

$$Bx+\{C+(AC-B^2)\,u\}\,x'=0, \quad \ldots\ldots(14)$$

and it is easy to verify that the equations (13) and (14) are consistent.*

To complete the identification, we write

$$z=u+C/(AC-B^2), \quad z'=v+A/(AC-B^2), \quad \ldots\ldots(15)$$

and

$$a=-B/(AC-B^2).$$

Then, from (14) and the corresponding formula containing y, y', we see that

$$x'=ax/z, \quad y'=ay/z\,;$$

and the relation between u, v gives also

$$z'=a^2/z.$$

These are of the same type as (4), *except that in the optical system* † $c=a^2$.

§ 3. **A simple example of the five aberrations of the first order.**

To illustrate the further use of the formulae found above in § 2, we shall consider these five aberrations in a particularly easy case: we shall suppose that the object consists of a bright spot "*at infinity*" so that the image is formed (according to the first approximation) on the second focal plane of the instrument.‡ To simplify the work, we take the second focal plane as a plane of reference: the first plane of reference is not specially chosen, but (for the sake of definiteness of thought) it may be looked upon as (for example) the first refracting surface of the lens-system.

Thus $v=0$ corresponds to making $u\to\infty$; and so we must have $A=0$: on the other hand, C depends on the choice of the first plane of reference, while $B=K$, the *power of the system*, which is the same for all planes of reference. To see this property, we need only note that, from equations (12)-(15), the focal length is $F=B/(AC-B^2)$; and so, when $A=0$,

$$B=-1/F=K,$$

adopting the convention that *the focal length is positive for convergent systems*.

Take the plane of zx to pass through the bright spot (forming the object); then we have

$$l=\sin\theta, \quad m=0, \quad \ldots\ldots(16)$$

if θ is the angle between the incident rays and the axis of the instrument. Then from the first pair of equations (10), we have the first approximations:

$$\left.\begin{aligned}l+K\xi'&=0, \quad 0+K\eta'=0,\\ \xi'&=Fl, \quad\quad \eta'=0.\end{aligned}\right\} \quad \ldots\ldots(17)$$

or

* In fact, if we eliminate the ratio $x':x$ we find that, for consistency, we must have

$$\{C+(AC-B^2)\,u\}\,\{A+(AC-B^2)\,v\}-B^2=0\,;$$

and if we multiply the first equation of (13) by $(AC-B^2)$, it will be found to agree with the equation just found. When $AC=B^2$, we obtain the telescopic case; this presents no fresh point of difficulty, but details are left to the reader.

† In general, the reader will have no difficulty in seeing that $c=a^2\mu'/\mu$, if the refractive indices are taken as μ, μ': and that, in the usual phraseology of Optics, this is equivalent to the statement that the two focal lengths are in the same ratio as $\mu:\mu'$. In the present case $-a$ is the focal length, as ordinarily defined; while, allowing for μ and μ', $-a$ is the *first* focal length and $-a\mu'/\mu$ is the *second*, c being equal to their product.

‡ Thus our problem corresponds to the image of a star in a telescope; or (for ordinary purposes) to the image formed on a plate by a photographic lens.

The terms of the second degree in ϕ may be denoted by

$$\tfrac{1}{2}(a, b, c, f, g, h)(X, Y, Z)^2,$$

and then, writing $A=0$, $B=K$, we have

$$\left.\begin{aligned}\phi_1 &= 0+aX+hY+gZ\\ \phi_2 &= K+hX+bY+fZ\end{aligned}\right\} \quad \ldots\ldots(18)$$

which are the forms to be used in (8). When the values (18) are substituted, the terms in X, Y, Z will lead to magnitudes of the third order; thus we can simplify X, Y, Z by using (17), and then we have

$$X=\xi^2+\eta^2, \quad Y=Fl\xi, \quad Z=F^2l^2 \ldots\ldots(19)$$

If we make use of (18) and (19), and then substitute in (8), we find that

$$\left.\begin{aligned}\xi' &= Fl+2F\xi\{a(\xi^2+\eta^2)+h(Fl\xi)+g(F^2l^2)\}\\ &\quad +F^2l\{h(\xi^2+\eta^2)+b(Fl\xi)+f(F^2l^2)\}\\ \eta' &= 2F\eta\{a(\xi^2+\eta^2)+h(Fl\xi)+g(F^2l^2)\}\end{aligned}\right\} \quad \ldots\ldots(20)$$

It will now be seen, from (20), that there are *five and only five independent coefficients* (a, b, f, g, h). There are *two* corrections which can be found completely, *without using skew rays*; namely, the *spherical aberration* (from the terms containing a) and the *distortion* from the term in f). But the remaining *three* corrections (since they involve terms such as $l\eta^2$ and $l^2\eta$) *cannot* be so determined; the names generally used are *coma* (from the terms in h) and *astigmatism and curvature* (from the terms in h and g, combined in a special way). For a fuller description of these effects the readers should consult ch. iii. (§§ 1-7) of Dr. Steward's *Tract.** T. J. I'A. BROMWICH.

914. [L¹. 5. b.] *Note on a Certain Locus connected with a Conic.*

The following problem is taken from Askwith's *Analytical Geometry of the Conic Sections*, page 180, Ex. 21:

If the normals at two points on the ellipse $x^2/a^2+y^2/b^2=1$ meet on the curve, prove that the tangents at these points meet on

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{(a^2-b^2)^2}{a^4b^4}\left\{(a^2-x^2)^2\frac{y^2}{b^2}+(b^2-y^2)^2\frac{x^2}{a^2}\right\} \quad \ldots\ldots(1)$$

The straightforward method, so dear to the heart of the novice, would be to eliminate θ and ϕ from

$$x\cos\theta/a+y\sin\theta/b=1; \quad x\cos\phi/a+y\sin\phi/b=1$$

and $$b^4\cos^2\theta\cos^2\phi\frac{\cos^2(\theta+\phi)/2}{\cos^2(\theta-\phi)/2}+(a^4\sin^2\theta\sin^2\phi\frac{\sin^2(\theta+\phi)/2}{\cos^2(\theta-\phi)/2}=\frac{a^4b^4}{(a^2-b^2)^2} \quad (2)$$

the last equation being the condition that the normals at θ and ϕ meet on the curve. On carrying out the elimination one obtains the sextic locus (1) given in the answer.

There is however a much neater method. Let the feet of the normals from P a point on the curve be N_1, N_2 and N_3, and let (x_1, y_1) be the pole of N_1N_2 so that (x_1, y_1) is a point on the required locus. The equation of N_1N_2 is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1,$$

and hence the equation of the chord joining the feet of the other two normals which are concurrent with those at N_1, N_2, is

$$x/x_1+y/y_1=-1 \quad \ldots\ldots(3)$$

(*Vide* Askwith, page 166.)

This is the line PN_3; hence applying the condition that (3) may be a normal to the given ellipse, we obtain the locus of (x_1, y_1) to be the conic

$$a^2x_1^2+b^2y_1^2=(a^2-b^2)^2 \quad \ldots\ldots(4)$$

* Review, p. 309.

That the locus of (x_1, y_1) is a conic and not a sextic may be deduced also from binary considerations. The triad of points N_1, N_2, N_3 is such that any one of them determines the other two and hence belong to a cubic involution I^3_1. The parameters of such a triad are, therefore, roots of cubics belonging to a definite pencil. On the other hand, we know that if there exists one triangle inscribed in the given conic S and circumscribed to another conic Σ, there exist an infinity of such triangles, and the parameters of their vertices are roots of cubics belonging to a pencil. Now the totality of cubic-pencils forms a manifold of the same order as straight lines in ordinary space, *i.e.* ∞^4, and this is also the order of the manifold of conics Σ which admit of triangles circumscribed to them and inscribed in S. Hence we deduce that *in any pencil of cubics the roots of any member represent the vertices of a triangle inscribed in S and circumscribed to another conic.*

It follows therefore that the chords N_1N_2, N_2N_3, N_3N_1 are all tangents to a definite conic Σ. (*Vide* Askwith, page 182, Example 33.) The locus of (x_1, y_1) is the polar reciprocal of Σ with respect to S.

We may expect the sextic (1) to contain the equation (4) as a factor. On dividing out we find the other factor to be

$$a^6y^2 + b^6x^2 = (a^2 - b^2)^2x^2y^2, \qquad \ldots\ldots(5)$$

which represents a rational quartic curve.

It remains to find out what this locus represents and why it figures at all in the answer. When we applied the condition that the intersection of the normals at θ and ϕ meet on the curve, we did not distinguish between the two possible cases that arise, viz. the point of intersection may be a point distinct from θ and ϕ or may be one of them. It is the latter hypothesis that is covered by the second method and gives the conic locus (4). In the former case the line joining the points θ, ϕ is itself a normal to the curve, and the required locus is now the locus of the poles of normal chords of the ellipse. This is the quartic (5), (*vide* Askwith, page 182, Ex. 37).

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915. [X. 1.] Cf. *Note* 904, *Gazette*, July 1928.

Johann Tourmayer Aventinus (1476-1534), a Bavarian historian, found at Ratisbon certain tables showing how the Romans counted on their fingers—a custom current in Italy in his day. Of these he published an account entitled *Numerandi per digitos manusque, quinetiam loquendi, veterum consuetudinis Abacus.* 4to. 1523.

916. [A. 1. c.] *Note on scheme for writing down Binomial Coefficients.*

In the expansion of $(1+x)^{-n}$ it is easy to show that

$$|\ {}_{-n}C_{r+1}\ | + |\ {}_{-n-1}C_r\ | = |\ {}_{-n-1}C_{r+1}\ |\ .$$

This leads to the following scheme for writing down Binomial Coefficients for a negative index:

$$\begin{array}{lccccc}
(1-x)^{-1} & 1 & 1 & 1 & 1 \ldots \\
(1-x)^{-2} & 1 & 2 & 3 & 4 \ldots \\
(1-x)^{-3} & 1 & 3 & 6 & 10 \ldots \\
 & & & \downarrow & \\
(1-x)^{-4} & 1 & 4 & 10 & 20 \ldots
\end{array}$$

The coefficients are read off horizontally or vertically, and written down by adding diagonally as shown by the arrow. This, of course, is the Arithmetical Triangle of Pascal in which marginal terms give binomial coefficients for a positive index. It also becomes clear that the pyramidal scheme for writing

down coefficients for a positive index may be used for coefficients when the index is negative as follows:

$$\begin{array}{lccccc} & (1-x)^{-1} & (1-x)^{-2} & (1-x)^{-3} & (1-x)^{-4} \\ & 1 & & & \\ (1+x)^1 & 1 & 1 & & \\ (1+x)^2 & 1 & 2 & 1 & \\ (1+x)^3 & 1 & 3 \rightarrow & 3 & 1 \\ & & & \downarrow & \\ (1+x)^4 & 1 & 4 & 6 & 4 \quad 1 \end{array}$$

The identity of the two schemes is seen by displacing all the columns in the second scheme, so that they commence on the first horizontal row.

The scheme can be used to write down coefficients for index $\frac{1}{q} \pm N$ where coefficients for index $\frac{1}{q}$ are known, when $N = 1, 2, 3$, etc.

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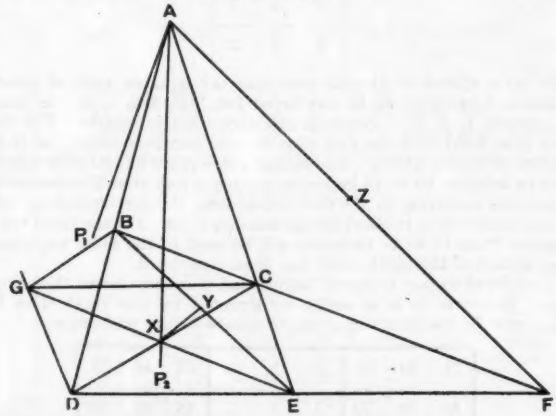
917. [K¹. 8. a.] *A short proof of three propositions.*



An elementary combined proof of the three propositions:

(*a*) the orthocentres of the four triangles formed by four straight lines are collinear;

(*b*) the middle points of the three diagonals of a complete quadrilateral are collinear;

(*c*) these lines are perpendicular to one another.

The four triangles formed by *ABD*, *ACE*, *FCB*, *FED* are

ABC, *ADE*, *FCE* and *FBD*.

The orthocentres are P_1, P_2, P_3 and P_4.

The quadrilateral is *BCED* and the middle points of the diagonals are *X*, *Y* and *Z*.

Draw *CG* equal and parallel to *ED*.

Then $AP_1 = BC \cot A$ and $AP_2 = DE \cot A$;

$$\therefore \frac{AP_1}{AP_2} = \frac{BC}{DE} = \frac{BC}{GC}.$$

Also the angle P_1AP_2 = the angle BCG;

$\therefore$ P_1P_2 is perpendicular to BG and therefore also to XY.

Similarly P_3P_4 is perpendicular to XY.

Similarly P_1P_3 and P_2P_4 are perpendicular to XZ, and P_1P_4 and P_2P_3 are perpendicular to YZ.

But four points cannot be joined by three pairs of parallel lines unless they are collinear.

Therefore $P_1P_2P_3P_4$ are collinear, XYZ are collinear and these lines are perpendicular to one another.

S. T. SHOVELTON.
A. M. PERRY.

918. [X. 10. a.] *Note on a Magic Square.*

After giving a lecture on the subject of magic squares, the following interesting construction for a magic square of the ninth order was suggested to me. I had not previously seen the method, and do not know whether it is new: in any case it may be of interest to some of your readers.

It is well known that there exists only one magic square of the third order

8	1	6
3	5	7
4	9	2

Divide up a square of 81 cells into nine sub-squares, each of order three. These nine sub-squares can be numbered 1st, 2nd, 3rd, ... in the same order as the integers 1, 2, 3, ... occur in the above magic square. The first sub-square is then filled with the first nine natural numbers exactly as they occur in the original square above. The second sub-square has its cells filled by the consecutive integers 10 to 18 inclusive in such a way that if x denote any one of the integers occurring in the first sub-square, the corresponding cell in the second sub-square is to be filled by the number $x+9$. For the third sub-square the numbers from 19 to 27 inclusive will be used in the same way, and so on until the square of the ninth order has been completed.

The completed square is shown below, and it is easy to see that the square is magic. Moreover it is as easily constructed by this method as by de la Loubère's rule for constructing a magic square of any odd order.

71	64	69	8	1	6	53	46	51
66	68	70	3	5	7	48	50	52
67	72	65	4	9	2	49	54	47
26	19	24	44	37	42	62	55	60
21	23	25	39	41	43	57	59	61
22	27	20	40	45	38	58	63	56
35	28	33	80	73	78	17	10	15
30	32	34	75	77	79	12	14	16
31	36	29	76	81	74	13	18	11

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919. [I. 3.] *A method of solving* $ax+by+cz=0$.

In the solution of equations with undetermined coefficients the following method of analysis may sometimes be profitably used.

Let it be required to find x, y and z to satisfy

$$106 \,.\, x+69 \,.\, y=163 \,.\, z=N \text{ say.}$$

Let R denote remainder after division of N by 163, the corresponding quotient being called q.

				After division by 163.			
	x	y	N	R	q	z	
(1)	1	1	175	12	1	—	
(2)	1	2	244	81	1	—	
(3)	1	3	313	−13	2	—	
(4)	2	4	488	−1	3	—	by adding lines (1) and (3)
(5)	24	48	5856	−12	36	—	by multiplying by 12
(6)	25	49	6031	0	37 =	37	by adding lines (1) and (5)

If, now, we are not satisfied with this result, we can proceed—first multiply (6) by (7):

(7)	175	343	42217	0	—	259	now reduce { x by 163 ; z by 106 }
(8)	12	343	24939	0	—	153	again reduce { y by 2×163 ; z by 2×69 }
(9)	12	17	2445	0	—	15	
(10)	24	34	4890	0	—	30	by doubling
(11)	1	15	1141	0	—	7	which is (6) less (10)

I have avoided division in order to make the method as universal as possible.

Another example:

$$5x+1521y=2354z.$$

				After division by 2354.			
	x	y	N	R	q	z	
(1)	1	1	1526	1526	0	—	
(2)	1	2	3047	693	1	—	
(3)	1	3	4568	−140	2	—	
(4)	5	15	22840	−700	10	—	
(5)	6	17	25887	−7	11	—	
(6)	120	340	517740	−140	220	—	
(7)	119	337	513172	0	218 =	218	
(8)	2380	6740	10263440	0	—	4360	
(9)	26	6740	10251670	0	—	4355	*i.e.* less 2354×5
(10)	26	2032	3090802	0	—	1313	*i.e.* less $2\times2354\times1521$
(11)	130	10160	15454010	0	—	6565	
(12)	11	9823	14940838	0	—	6347	*i.e.* (11) less (7)

Now divide by 11:

	1	893	1358258	0	—	577	

This is possible because each of the x, y and z is so divisible.

H. J. WOODALL.

920. [V. 1. 9.] *A Chinese Money Changer.*

At the money-changers in the Blue Town, to which we went to sell some silver, the Chinese dealers essayed, according to custom, to apply this fraud to us, but they were disconcerted. The weight shown by their scales was perfectly correct, and the price they offered us was rather above the ordinary course of exchange, and the bargain between us was so far concluded. The chief clerk took the *suan-p'an*, the calculation table used by the Chinese, and

after calculating with an appearance of intense nicety, announced the result of his operation. "This is an exchange-office," said we; "you are the buyers we the sellers; you have made your calculations, we will make ours: give us a pencil and a piece of paper."—"Nothing can be more just; you have enunciated a fundamental law of commerce," and so saying they handed us a writing-case. We took the pencil and a very short calculation exhibited a difference in our favour of a thousand sapeks. "Superintendent of the bank," said we, "you *suan-p'an* is in error by a thousand sapeks."—"Impossible! Do you think that all of a sudden I've forgotten my *suan-p'an*? Let me go over it again"; and he proceeded with an air of great anxiety to appear correct, to set his calculating machine once more in operation, the other customers by our side looking on with great amazement at all this. When he had done: "Yes," said he, "I knew I was right; see, brother"; and he passed the machine to a colleague behind the counter, who went over his calculation; the result of their operations was exactly the same to a fraction. "You see," said the principal, "there is no error. How is it that our calculation does not agree with that which you have written down there?" "It is unimportant to inquire why your calculation does not agree with ours: this is certain, that your calculation is wrong and ours right. You see these little characters that we have traced on this paper; they are a very different thing from your *suan-p'an*; it is impossible for them to be wrong. Were all the calculators of the world to work the whole of their lives upon this operation, they could arrive at no other result than this; that your statement is wrong by a thousand sapeks."

The money-changers were extremely embarrassed, and began to turn very red, when a bystander, who perceived that the affair was assuming an awkward aspect, presented himself as umpire. "I'll reckon it up for you," said he, and taking the *suan-p'an*, his calculation agreed with ours. The superintendent of the bank hereupon made us a profound bow: "Sirs Lamas," said he, "your mathematics are better than mine." "Oh, not at all," replied we, with a bow equally profound; "your *suan-p'an* is excellent, but who ever heard of a calculator always exempt from error? People like you may very well be mistaken once and a way, whereas poor simple folks like us make blunders ten thousand times. Now, however, we have fortunately concurred in our reckoning, thanks to the pains you have taken. . . ."

After our conciliatory address had restored self-possession to all present, everybody drew round the piece of paper on which we had cast up our sum in Arabic numerals. "That is a fine *suan-p'an*," said one to another; "simple, sure, and speedy." "Sirs Lamas," asked the principal, "what do these characters mean? What *suan-p'an* is this?" "This *suan-p'an* is infallible," returned we; "the characters are those which the Mandarins of Celestial Literature use in calculating eclipses, and the course of the seasons." After a brief conversation on the merits of the Arabic numerals, the cashier handed us the full amount of sapeks, and we parted good friends.—Huc and Gabet: *Travels in Tartary, Thibet and China*, 1844-6 (Reprinted London, 1928, vol. i. p. 141). [Per Mr. F. PURYER WHITE.]

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640. The following rule for [extracting the cube root] will be found much easier to be remembered than the common method, and far more exact and expeditious.

Rule.—Twice the nearest cube, added to the given number, is to the difference between the given number and the nearest cube, as the root of the nearest cube, to the correction; which must be ADDED to the root of the nearest cube, if the nearest cube is less than the given number, otherwise SUBTRACTED. —Preface to *The Theory of Gunnery*, Reuben Burrow, 1779.

REVIEWS.

The Symmetrical Optical System [Cambridge Tracts, No. 25] by G. C. STEWARD. Pp. viii + 102. 7s. 6d. net. 1928. (Cambridge University Press.)

Those who have (like myself), spent some considerable time in trying to obtain an easy correlation between elementary Geometrical Optics and the formulae used for the practical computation of von Seidel's five aberrations will be very grateful to Dr. Steward for having published this Tract. The fundamental importance of these aberrations is known to all who have to deal with the actual design of modern lenses: without the patient labours* of the mathematicians interested in this department, there would be no high-speed photography possible, and consequently the colossal industry of "*moving pictures*" depends quite as much on the less picturesque work of this type as on that of "*film-stars*" at Hollywood.

The treatment depends on the use of Hamilton's Characteristic Function and its modification by Bruns into the Eikonal: their general properties are first investigated (in Ch. II), where some reference might have been given to Sir Joseph Larmor's papers and to some later ones in the *Proceedings of the London Mathematical Society*.† And in Ch. III will be found an investigation (based on Lord Rayleigh's) which establishes the existence of five (and only five) "*first order aberrations*" ‡‡ for a symmetrical optical system; but in this connexion it seems right to remark here that Sir Joseph Larmor had given an equivalent investigation in lectures, at least five years (1907) before Lord Rayleigh's paper was published.§ The practical methods of computation are explained by Dr. Steward in Ch. V. of this tract.

The connexion between Mr. T. Smith's recent (1923) "*Optical Cosine Law*" and the older "*sine-condition*" (first given by Abbe in 1873, and connected with the characteristic function by C. Hockin in 1884) is carried out in Ch. IV; the same chapter contains also Dr. Steward's own method of connecting Herschel's condition (for spherical aberration) with the same cosine-law.‖ Finally the last two chapters (VI and VII) are devoted to the diffraction patterns, due to the various aberrations and to various types of aperture: during recent years Dr. Steward has published much original work in this direction, and it is extremely useful to have this summary of his work (and that of earlier writers) collected in a readable whole.

I venture, however, to criticise one defect as it seems to me in the Elementary Theory (Ch. I): this part of the subject has fallen very frequently to my lot in College Lectures, and I have never felt that any discussion could be adequate which ignores skew rays.¶ The method followed in more than one text-book is to give a full account for rays in a plane with the axis of symmetry, and then *to assume without proof* that the projection of a skew ray (on a plane through the axis) *must* behave as if it were itself an actual ray. When this assumption is put into *words*, I have often found that it strikes average Cambridge students as *unlikely* to be true. Since the result is an *approximation* only, and is *not* exact, it is hardly satisfactory to appeal to intuition; but the proof by means of direction-cosines and coordinates, following Gauss, is quite easy

* The Goerz "Dagor" lens (the first of the modern photographic anastigmats) is reputed to have taken five years to design; and it is said that the royalties amounted to £100,000 at least.

† See vols. 20, 23, 29 (R. A. Sampson), 31 (T. J. I'A. Bromwich) in the First Series of *Proceedings*. Prof. Sampson has published a connected account of aberrations, as affecting astronomical instruments, in the *Philosophical Transactions*, and this might also have been quoted by Dr. Steward, as well as investigations by Prof. L. N. G. Filon and Cdr. Baker.

‡‡ These aberrations are of the *third* order in the small magnitudes which define the incident pencil of light: the term *first* implies that (amongst the various corrections) possible this group occurs before any others.

§ A slight variation is also to be found in Dr. Leathem's Tract (No. 8 of the same series) on *The Symmetrical Optical Instrument*.

‖ The practical importance of the sine-condition for the design of microscopes is well known; it is perhaps less familiar that the older designs, found by the patient "trial and error" process do, as a matter of fact, satisfy the sine-condition, although this condition was not known to their makers. Herschel's condition is equally important for telescopes.

¶ That is, rays which do *not* intersect the axis of the symmetrical instrument.

and convincing.* The discussion by Dr. Steward, however, depends on the method of one-to-one correspondence (not tracing out actual ray-paths); but it suffers from a defect similar to that just mentioned: and I do not think that the treatment can be held to be complete without some analytical developments similar to those given by Drude in his *Optics*.

I should have liked also to see more use made of the wave-theory in the sense that all Geometrical Optics is contained in the principle of the stationary reduced path; this amounts to regarding the wave-length as tending to the limit zero, instead of being minute but finite. This principle leads almost intuitively to the fact that if P, Q are conjugate foci (using the customary phrase), then the reduced path from P to Q must be the *same* for *all* directions of the ray at P. Having obtained this property we can predict at once the existence of the *aplanatic points* (Ch. I § 14); and that these are the only possible *exact foci* for refraction at a spherical interface; and further that the form of surface which gives P, Q as *exact* foci is in general derived by the rotation (about PQ) of a Cartesian oval having P, Q as (geometrical) foci. This last fact is really the justification for the use of the term *spherical aberration*, which is really due to the deviation of the sphere from the more accurate surface. Dr. Steward does apply (Ch. II § 3) the method sketched here in order to obtain the relation $f_1/f_2=\mu_1/\mu_2$ between the two focal lengths (f_1, f_2) of the symmetrical instrument: this investigation is attributed to Pelletan, a reference which I have not verified. For my own part I learned the method from Sir Joseph Larmor's lectures (1893-4), who referred us to Clerk Maxwell's paper in the *Proceedings of the London Mathematical Society* (vol. 2).

In conclusion, I welcome the large number of carefully drawn diagrams, which illustrate the effects due to the different aberrations (21 typical diagrams are given in Ch. III), and also diffraction effects (6 typical intensity-charts are drawn in Chs. VI and VII). Without such drawings it is almost impossible to visualise the meaning of the formulae; in this respect I can sympathise both with students and teachers who have had to use existing text-books, which leave the reader to provide his own diagrams—or else go without.

T. J. I'A. BROMWICH.

Vorlesungen uber Variationsrechnung. By KARL WEIERSTRASS. Bearbeitet von Rudolf Rothe (Mathematische Werke von Karl Weierstrass herausgeg. unt. Mitwirkung einer von der Preussischen Akademie der Wissenschaften eingesetzten Kommission 7*ter* Bd.). Pp. viii + 324, 8vo. 45 Rm. 1927. (Leipzig, Akad. Verlagsgesellschaft).

Calculus of Variations. By A. R. FORSYTH. Pp. xxii + 656, 8vo, £2. 1927. (Cambridge University Press.)

The progress of K. Weierstrass in the Calculus of Variations was made in two essentially different ways, by amending the work of his predecessors in the field, and by introducing and utilizing new concepts and new methods.

In his earlier work, prior to the year 1879, he succeeded in removing all the difficulties that were contained in the old investigations of Euler, Lagrange, Legendre, and Jacobi, simply by stating precisely and analysing carefully the problems involved. In improving upon the work of these men he did several things of paramount importance. The chief of these may be summarized as follows:

(1) he showed the advantages of parametric representation;
(2) he pointed out the necessity of first defining in any treatment of a problem in the Calculus of Variations the class of curves in which the minimizing curve is to be sought, and of subsequently choosing the curves of variation so that they always belong to this class;
(3) he insisted upon the necessity of proving carefully a fact that had hitherto been assumed obvious,† *i.e.*, that the first variation does not

* A simple treatment will be found in this number of the *Gazette* (p. 301).

† It is true that the first proof of the fundamental Lemma of the calculus of variations had been given by Stegemann (1854) and that Weierstrass himself quotes Heyne. But Weierstrass was probably the first who explained clearly the reasons why such a proof is necessary.

always vanish unless the differential equation, which is now called the "Euler Equation," is satisfied at all points of the minimizing arc at which the direction of the tangent varies continuously;

(4) he made a very careful study of the second variation and proved for the first time that the condition $\delta^2 I > 0$ is sufficient for the existence of a weak minimum.

2. The second part of the work of Weierstrass is directly related to his concept of a *strong* minimum. It is said nowadays that a minimum is "weak" provided that it holds only when the minimizing curve is compared with those curves whose directions at corresponding points differ little from the direction of the minimizing curve. A minimum is "strong" provided that it holds when the minimizing curve is compared with those curves which remain in its vicinity, no restriction being made as to direction.

Weierstrass found very early that it is essential to consider the strong minimum as well as the weak, but he became convinced during his research that the classical methods were inadequate for handling it. In 1879 he discovered his E-function and with it was able to establish conditions sufficient for the existence of a strong minimum.

3. Although Weierstrass published none of his work, one is able to follow his various steps in sets of notes taken by students who at different times attended his lectures on the subject. These notes, although interesting to the person who wishes to follow the trend of Weierstrass' ideas, yield, unless considerable effort is made, an incomplete and inaccurate account of the development, and I am convinced that the editor of the seventh volume of Weierstrass' Works is justified in taking the position that one should have at hand the whole of the work of Weierstrass in the Calculus of Variations, organized and in a single volume. To this end he has collected material from all available sources, and his efforts have resulted in a book which, except for its price, is attractive in every way. It is one of the best elementary text-books I know on the subject.

The first few chapters of the book contain the theory of ordinary maxima and minima and the transformation of quadratic forms. The intermediate chapters contain a complete treatment of the ordinary and isoperimetrical problem in the plane, and deal with the older theory of the second variation as well as the theory concerning the E-function. The last chapter is concerned with problems which are less generally treated and involve one-sided variations. Here is found Weierstrass' solution of some geometrical problems solved in answer to the challenge of Steiner, who was of the opinion that his methods of pure geometry could not be replaced by the analytic methods of Weierstrass.

On the whole, the book is excellent for the student interested in learning the fundamentals of the subject, the contents being adequate and the presentation clear and rigorous. In fact, the only unfavourable criticism I have of the book as an elementary text-book is that a much simpler exposition of the material in the first few chapters can be made by using some of the modern developments. Weierstrass' methods were the best of his time but seem cumbersome now. On the contrary, however, the scholar who is acquainted with the modern developments in the Calculus of Variations will find no new facts in this book, which is published much too late to exert any influence of importance on the advancement of this part of analysis; nearly everything of value in the work of Weierstrass has been published several times, as for example in the admirable books of Kneser (1900) and Bolza (1904 and 1909). We must be content with the fact that we now possess, after nearly a half-century, the first authentic exposition of a theory which forms the basis of all progress in the Calculus of Variations.

4. I wish to state in the beginning of my review of the second book I have to consider, Professor Forsyth's *Calculus of Variations*, that this volume is clearly the result of a tremendous amount of work. Professor Forsyth not only treats ordinary and isoperimetrical problems in the Calculus of Variations in the plane and space, but he studies also cases in which the integrand involves second derivatives, and applies his theory to problems which involve double and triple integrals. He uses the ordinary and parametric representation, and illustrates his work with many problems in geometry, mechanics and optics.

He does, however, omit apparently intentionally all material recently developed in the subject, his purpose being to obtain among other things all the results of Weierstrass, in modifying conveniently the methods of the 18th and early 19th centuries.

5. Several sections of the book interested me considerably, but it is not entirely satisfactory. Some of the proofs are not convincing and the author is not altogether free from the danger of making mistakes. To illustrate the latter, I cite particularly the material in Paragraph 72, because it is characteristic of his reasoning and is as easy to show by an example that his conclusions are not warranted. There he discusses in a general way the possibility of joining two points A and B of the plane by an extremal (or "characteristic" curve as he calls it). He apparently believes that this problem can be solved by a succession of "finite, though small," approximations, provided there are no points conjugate to A on the portion of the curves he considers successively, thus overlooking entirely the possibility that his successive approximations though remaining *finite* may in the course of time become so *small* that the point B is never attained by any of the curves under consideration. Consider the case in which $F(x, y, \dot{x}, \dot{y}) = e^x(\dot{x}^2+\dot{y}^2)^{\frac{1}{2}}$ and A, B are the points (0, 0) and (0, 4) respectively. Here the extremals either are of the form $y = y_0$ or are given by the equation $ex - x_0 \cos(y - y_0) = 1$, and so the y-coordinates of any two points on any extremal differ by less than π. Hence no extremal joins A and B and yet on no curve passing through A is there a point conjugate to A.

6. Minor errors of this kind which are easily detected, and even those which are apt to call forth the skill of trained mathematicians, would never in my opinion handicap a book of such importance as Professor Forsyth's. Unfortunately there are other faults of a more serious nature—lack of proper development in particular—and these involve the most essential parts of the theory. I cite two instances:

(*a*) The author defines in the last few lines of page 9 a "regular" function as one which with its derivative is uniform, finite, and continuous. He assumes on page 13 that the variation ν is a regular uniform function of x within the interval of integration. He then supposes, on page 14, ν positive in this interval and zero elsewhere, and makes similar suppositions in nearly every chapter of the book. Now it seems that under these conditions the author might have devoted a few of his 650 pages to showing how a regular function of this kind can be constructed. Such a discussion seems even more necessary when one considers that this construction involves the consideration of functions with such high orders of singularities as e^{-1/x^2} has at the origin.

(*b*) The author is quite satisfied when he has shown under what conditions the second variation is positive in the various problems he treats, because as he says when the first variation is made to vanish the quadratic terms become "dominant." But he never discusses the question of the existence of a finite range of curves in the neighbourhood of the extremal under consideration for which the sign of the variation is dependent only upon these quadratic terms—a question which Weierstrass examined with great pains.

Omissions of this sort in my mind impair the value of the book much more than occasional mistakes. It would certainly be impossible for a person to learn the theory of the Calculus of Variations using only this book of Professor Forsyth's, unless he had an unusual degree of mathematical ability. I might point out here that on the other hand one who is thoroughly acquainted with the modern development of the subject could read many parts of the book with interest and profit.

7. I must point out also that in his treatment of multiple integrals the author does not even mention the most important and most simple boundary conditions for the extremals. Thus, for example, in spite of the fact that he introduces problems on minimal surfaces, he does not mention the problem of Plateau, to which Schwarz devoted most of his work on minimal surfaces. It is true that other methods than those used by the author are necessary to treat the latter problem, and that a satisfactory solution of reasonable generality has been obtained only in recent years, but it does seem that something might be said in regard to a problem which has been a subject of interest

to mathematicians for many years, in the largest book ever written in English on the Calculus of Variations.

C. CARATHÉODORY.

Mathematical Analysis. (Higher Course). By F. L. GRIFFIN. Pp. x+512. 8s. 6d. 1928. (Harrap & Co.)

Prof. Griffin's volume has such an imposing title that the reviewer (who knows his limitations) opened it in fear and trembling expecting to find it quite out of the range of his knowledge. But this is an American book, and university education in America appears to begin immediately after the standard of matriculation has been reached, so that the first two years of the university course correspond to the two years' advanced course in an English secondary school. Consequently the reviewer's fears vanished.

It is true that the study of mathematical rigour belongs to the stage of what the English regard as university education, but it is disturbing to find in a book, written for second-year American university students, the following remark on the Fundamental Theorem of the Integral Calculus: "This theorem can be proved rigorously by a purely algebraic argument of some length; but it is most easily seen geometrically." Most boys of seventeen or eighteen, who have been properly grounded, can grasp the purely algebraic argument and find it interesting. In the discussion on "Taylor Series" also more might have been done; there is no indication, for instance, that Taylor's theorem is an extention of the theorem

$$f(b)=f(a)+(b-a)f'(\xi), \quad a<\xi<b.$$

In the appendix there is a discussion on a double integral as a limit of a sum, and the above "difficulties" might well have found a place there if it was thought undesirable to include them in the body of the book. The appendix also contains the Greek alphabet and a large collection of standard formulas (*sic*). One wonders what the American plural of the word octopus would be.

While, in the reviewer's opinion, not enough stress is laid on the logic of the subject, a great deal of ground is covered. There is an enormous "sandwich" on double and treble integrals, the "meat" consisting of a long section on coordinates in space—quite a little treatise on solid geometry. The author further makes a big noise about mean values and approximate integration, but, by way of illustration, stops short at Simpson's rule. There is a good chapter on simple differential equations, and, surprisingly enough, near the end of the book a chapter on the conics. The book is thus an *omnium gatherum* of elementary pure mathematics, no doubt suited to the needs of the students of Reed College for whom it was written. But, perhaps by intention, it is superficial.

The publishers have done their work well. The book is printed clearly on good paper, and it lies open flat wherever it is opened. I have not found any misprints, and there is a very full index. The price is wonderfully moderate—five pages for a penny.

Calculus. By H. B. FINE. Pp. viii+421. 14s. net. 1927. (The Macmillan Company.)

This book is twice as expensive as the last: its cost works out at five pages for twopence, and it is worth it. It, too, is well got up and has an index. It is a serious book and difficulties are not burked. It consists of thirty chapters, the first twenty being a textbook of elementary calculus containing the material usually given in a one-year course. "Except for brief chapters on determinants and solid geometry the remainder of the book is devoted to topics of advanced calculus, including the theory of partial differentiation and some of its more important and interesting applications, differential equations, line and surface integrals, the development of implicit functions in series by the method of successive approximations with applications to curve tracing, Fourier series, and functions of a complex variable. . . . The proofs are based on the fundamental theorems respecting the existence of limits and the properties of continuous functions. These theorems are formulated and illus-

trated geometrically early in the book, but their arithmetical proofs are postponed to a chapter on numbers and continuous functions near the end." This chapter, the last but one, is particularly well done, and the whole of the second part is valuable as an introduction to the continental treatises. The book is therefore suited to the needs of scholarship candidates, or of undergraduates in their first year at Oxford or Cambridge. N. M. GIBBINS.

Frequency Curves and Correlation. By W. PALIN ELDERTON. Second Edition. Pp. vii + 239. 15s. net. 1927. (C. & E. Layton.)

Mr. Elderton's book, first published in 1906, was primarily intended for actuaries, but it is more widely known as a handy guide to Pearson's forms of equation of a frequency-distribution. It falls naturally into three portions: (I) Frequency-distributions; (II) Correlation; (III) Appendices dealing with miscellaneous points.

(I) The first portion deals mainly with the problem of selecting the particular type of Pearson formula which is applicable to the data, and with the method of determining the constants for the type chosen. This, in the first edition, occupied 58 pages out of the total 105 given to portion (I). In the new edition this is amplified by the inclusion of some uncommon types of distribution; this new matter is partly taken from an Addendum published in 1917. But there is also a new chapter dealing with other forms of distribution than those found by Pearson: in particular, with forms put forward by Edgeworth and Charlier. This is an extension of an Appendix in the first edition.

(II) In the second portion the chapter on correlation has been extended and broken up into two. The first deals with correlation generally, and with the calculation of the correlation coefficient r. The second deals with the equation to the joint frequency-distribution of two normally correlated variables, and with correlation of ratios. The chapters on correlation of characters not quantitatively measurable, on probable errors, on the test of goodness of fit, and on contingency, have been slightly extended; and a very brief chapter on partial correlation has been added.

(III) The Appendices now include one on Corrections for Moments, giving formulae for calculating the "corrections for abruptness." Some caution is necessary in using these formulae; the expressions given are for the terms at the start of the distribution, but the corresponding terms at the end of the distribution must apparently have the signs of alternate groups of terms altered, beginning in each case with the last group.

This Appendix contains a numerical example which purports to show that the value of the moment "with Sheppard's adjustment" is less accurate than the value "with full adjustment." With regard to this, the reviewer may perhaps be permitted to point out that the so-called "Sheppard's adjustment" comprises part only of the complete general formula (for ordinary cases) published by him in 1898 (ref. on p. 29 of the book), and that the Pairman-Pearson (1919) formulae quoted in the Appendix are merely particular formulae, of somewhat restricted application, which are easily deduced from the general formula. W. F. S.

Bell's Everyday Arithmetics. By J. B. THOMSON. Books I. and II., 8d. each; III. and IV., 10d. each. Teachers' Books with Answers, I. and II., 2s. each; III. and IV., 2s. 3d. each. 1928. (Bell.)

This series, covering notation, the first four rules, reduction, bills, weights and measures, vulgar and decimal fractions, averages, proportion, circles, ovals, plans and maps, looks like a very pleasant and interesting introduction to the mysteries of arithmetic. It is in particular notable for the large type in which numbers are printed, the realism that prevails throughout, and wide areas of everyday life from which the examples are drawn. It is well worth the attention of those responsible for the training of the first four Standards.

Fundamental Arithmetic. By P. B. BALLARD. Pupil's Book VII., pp. 113, 1s. 4d. Teacher's Book VII., pp. 87, 2s. 6d. 1927.

Arithmetical Dictation By A. Wisdom. Books 5 and 6, 1s. 1d. (paper cover); 1s. 3d. (limp cloth). 1927.

Teaching the Essentials of Arithmetic. By P. B. Ballard. Pp. xxiv +260. 6s. net. 1928.

The above volumes are all published by the University of London Press. The two booklets first on the list bring to a close the general preparation of their predecessors and prepare for the School Certificates, the London Matriculation, and the like. As an example of the systematic arrangement of the work we might draw attention to the questions under the heading of "The Diagnostic Test" (pp. 91-98). "I have endeavoured in this test to pack the maximum of ideas into the minimum of space. Each question has a point, and a point different from all the others. The whole field of study is covered by the 125 questions. . . . I have called the test 'diagnostic' because it not only measures the pupil's chance of passing the school certificates examination, but it reveals his points of weakness—the points on which he needs either enlightenment or practice."

Mr. Wisdom's contribution is a well-devised series of exercises on Mental Arithmetic, based on the Fundamental Arithmetic Books, and intended to fit in with, to revise and to extend the written work already done.

Dr. Ballard presents us in the last of these books with a profusion of excellent quotations as chapter headings. Perhaps a series of short quotations from his own chapters will give an idea of the stimulating manner in which he treats his subject:

Habit formation is . . . a process of emancipation (p. 7).

Habit is a servant; see that it is a good servant. Intelligence is a master; see that it is not allowed to concern itself too much with life below-stairs (p. 15).

The wise teacher will refuse to make a clean-cut choice between deductive and inductive methods: he will use both (p. 25).

One means of bringing [the joy of achievement] into the arithmetic lesson is to capture some of the spirit of sport (p. 33).

Every pupil in arithmetic has his own *tempo* (p. 42).

Any change that is proposed [in a traditional method] should be cautiously, if not suspiciously examined (p. 44).

The pitfalls are there and the pupils should be made acutely aware of their existence. Sometimes the only way to do this is to push the pupils in (p. 74).

There are little systems which, when they have had their day, should cease to be. The difficulty is to discern when they have had their day (p. 88).

The nought times is the table that most frequently breaks down in practice. . . . Even the once table is not so easy as it seems (p. 120).

If you use intelligent methods, check your result by other intelligent methods —or by the routine method (p. 138).

As a new method it ranks with that of the boy who had discovered a new way to count sheep: count their legs and divide by four (p. 149).

"At any rate it makes the pupils think" [they say]. It does. That is its most serious indictment. It wastes mental effort (p. 160).

We cannot afford to despise such things as recurring decimals without paying some sort of penalty (p. 199).

It may amuse the reader to try to "spot" the particular topics that suggested some of these remarks.

To teachers young and old who have to teach arithmetic this suggestive volume will be found to provide much food for thought.

Exercises in Arithmetic. By E. R. Pigrome. Parts I.-III., 1s. each, with Answers. 1926-1928. (Humphrey Milford, Oxf. Univ. Press.)

A useful and well-graded collection of examples, arranged on the concentric system, and designed for preparatory schools as a basis for the later work in lower middle forms of secondary schools.

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641. Le 7 septembre 1783 . . . Euler cessa de calculer et de vivre.—Condorcet, *Éloge de M. Euler*, Œuvres de Condorcet (edited by Arago), 1847, vol. 3, p. 41.

An Intermediate Arithmetic. By F. L. GRANT and A. M. HILL. Pp. viii +266. 4s. with Answers. 3s. 6d. without Answers. 1927. (Longmans, Green.)

This manual is devised for the benefit of "the first three forms of secondary schools," and for pupils in continuation central schools. The first part is notable for the attention paid to place values, the use of decimals, and the stress laid on checks and rough estimates. The second part is called "applied" arithmetic and covers the ground required for the usual examinations. The number of worked-out examples seems larger than usual, and in addition to ample mental and oral tests there is an over-generous supply of questions for solution. Integers throughout are in black type, but vulgar fractions, which are far more trying to eyes young and old, are left as of yore. Such misprints as we have noticed are few and trivial. The authors are evidently experienced teachers.

MATHEMATICAL ASSOCIATION

(LONDON BRANCH).

MEETING at Bedford College, 10th November, 1928. Mr. C. T. Daltry introduced a discussion on "What we should teach in Graphs." He considered that reading graphs should precede sketching, and sketching should precede plotting. He gave a number of examples to show that there were subjects, more topical and instructive than watered-down co-ordinate geometry. Graphs of functions could be used to show the development of mathematical ideas. There were about 70 members present, and Mr. Siddons, Dr. Sheppard, Prof. Lodge and Messrs. W. C. Fletcher, Katz and Boon continued the discussion.

Meeting at Bedford College on Saturday, 8th December, 1928. Mr. Katz read a paper on "How do we justify the place that Mathematics has hitherto held in the School Curriculum?" The Chairman (Mr. Fletcher), Dr. Sheppard, Miss King and Messrs. Taylor and Bickley joined in the discussion. At the close of the meeting Mr. Katz referred warmly to the interest Mr. Fletcher had taken in the activities of the Branch, and expressed the general regret that as he was leaving London he would not be able to attend meetings as regularly in the future.

F. C. BOON, Hon. Sec.

ERRATA.

P. 178, l. 23. *For* This slide *read* Slide II.

l. 3 up. Insert a comma after each square root.

P. 181, l. 10. *For* asymptote *read* asymptote—.

P. 183, l. 3 up. *For* here *read* on Slide II.

l. 8. *After* this figure *insert* (Slide IV.).

P. 184, l. 1. *After* gun *insert* (Slide III.).

Of the Plates: Slides I. and II. should face p. 178.

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